ETA-EINSTEIN CONDITION ON TWISTOR SPACES OF ODD-DIMENSIONAL RIEMANNIAN MANIFOLDS

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Abstract. In this note, we find the conditions on an odd-dimensional Riemannian manifolds under which its twistor space is eta-Einstein.

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1. Introduction

The construction of the twistor space of an odd-dimensional Riemannian manifold (as well as the one in the even-dimensional case) can be traced back to R. Penrose [9, 10]. These spaces have been studied by many people, mainly from the point of view of the CR-geometry (see, for example, the literature quoted in [5, 6, 7]).

It is convenient to modify slightly the construction in [8, 11] and to define the twistor space of an odd-dimensional Riemannian manifold (M, g) as the bundle $\mathcal{C}$ over M whose fibre at a point $p \in M$ consists of all pairs (φ, ξ) of a skew-symmetric endomorphism φ and a unit vector ξ of T_pM such that (φ, ξ, g_p) satisfies the (algebraic) identities in the definition of an almost contact metric structure ([5, 6]). The smooth manifold $\mathcal{C}$ admits two natural partially complex structures (f-structures) Φ_1 and Φ_2, a 1-parameter family of Riemannian metrics h_t, $t > 0$, compatible with Φ_1 and Φ_2, and a globally defined h_t-unit vector field χ such that (Φ_α, χ, h_t), $\alpha = 1, 2$, is an almost contact metric structure on $\mathcal{C}$ (we refer to [3] for general facts about such structures). According to [7] the metrics h_t on $\mathcal{C}$ are never Einstein. In this note we consider on $\mathcal{C}$ a generalization of the Einstein condition adapted for almost contact metric manifolds, namely the so-called eta-Einstein condition. Let us note that the almost contact metric structures (Φ_α, χ, h_t), $\alpha = 1, 2$, are never Sasakian (even they are never contact [5, 6]). The eta-Einstein condition for Sasakian manifolds has been recently discussed in [4].

Recall that an almost contact metric structure with contact form η and associated metric h on an odd-dimensional manifold N is said to be eta-Einstein if there exist smooth functions a and b on N such that

$$Ricci_h(X, Y) = ah(X, Y) + b\eta(X)\eta(Y), \;\; X, Y \in TN. \tag{1}$$

It is clear that the functions a and b are uniquely determined - denoting by s and ξ the scalar curvature of h and the h-dual vector field of η, we have $s = a(dim\,N) + b$, $Ricci(\xi,\xi) = a + b$. Note that, in contrast to the Einstein case, the functions a and b are not constant in the general case.

The main purpose of this paper is to prove the following

Theorem 1. *Let (M,g) be a Riemannian manifold of odd dimension $n \geq 3$. Then its twistor space $\mathcal{C}$ endowed with the metric h_t and the contact form $\eta_t = h_t(\cdot,\chi)$ is eta-Einstein if and only if $n = 3$, the manifold (M,g) is of positive constant curvature ν and $t\nu = \frac{1}{2}$; in this case we have $a = \dfrac{3\nu}{2}$ and $b = -\dfrac{\nu}{2}$ where the functions a and b on $\mathcal{C}$ are defined by means of* (1) .

The proof is based on a coordinate-free formula for the Ricci tensor of the metric h_t in terms of the curvature of the base manifold (M,g) obtained in [7].

Suppose that the base manifold M is oriented. Then its twistor space $\mathcal{C}$ is the disjoint union of the open subsets $\mathcal{C}_{\pm}$ consisting of the points (φ,ξ) that yield $\pm$ the orientation of T_pM via the decomposition $T_pM = Im\varphi \oplus \mathbb{R}\xi$ in which the vector space $Im\varphi$ is oriented by means of the complex structure φ on it. These open sets are diffeomorphic by the map $(\varphi,\xi) \to (\varphi,-\xi)$ which sends (Φ_α,χ,h_t) to $(\Phi_\alpha,-\chi,h_t)$. If, in addition, M is of dimension 3, the map $(\varphi,\xi) \to \xi$ is a diffeomorphism of $\mathcal{C}_{\pm}$ onto the unit tangent bundle T_1M of M which sends the characteristic vector field χ to the standard characteristic vector field on T_1M (but neither of the structures Φ_α is going to the standard partially complex structure of T_1M); this map sends also the metric h_t to the dilation of the Sasaki metric by $2t$ in the vertical directions. Thus Theorem 1 implies that the Sasaki metric on the unit tangent bundle of the 3-sphere endowed with the $+1$-curvature metric is eta-Einstein. In fact, by a result of S.Tanno [12], the Sasaki metric on the unit tangent bundle of any unit sphere S^m is eta-Einstein and a suitable modification of this metric gives a homogeneous Einstein metric on T_1S^m.

2. Preliminaries

Let V be a real n-dimensional vector space with a metric g. A partially complex structure on V (or f-structure) of rank $2k$ is an endomorphism F of V of rank $2k$, $0 < 2k \leq n$, satisfying $F^3 + F = 0$. We shall say that such a structure F is compatible with the metric g if the endomorphism F is skew-symmetric with respect to g.

Given a compatible partially complex structure F, we have the orthogonal decomposition $V = ImF \oplus KerF$ and F is a complex structure on the vector space ImF compatible with the restriction of g.

Denote by $F_k(V,g)$ the set of all compatible partially complex structures of rank $2k$ on (V,g). The group $O(V)$ of orthogonal transformations of

V acts transitively on $F_k(V,g)$ by conjugation and $F_k(V,g)$ can be identified with the homogeneous space $O(n)/(U(k)\times O(n-2k))$; in particular, $dim F_k(V,g) = 2nk-3k^2-k$. By the results of [1], the homogeneous manifold $F_k(V,g)$ admits a unique (up to homotety) invariant Kähler-Einstein structure structure $(h,\mathcal{J})$. It can be described in the following way (see, for example, [7, 5, 6]): Consider $F_k(V,g)$ as a (compact) submanifold of the vector space $so(V)$ of skew-symmetric endomorphisms of (V,g). Then the tangent space of $F_k(V,g)$ at a point F consists of all endomorphisms $Q\in so(V)$ such that $QF^2+FQF+F^2Q+Q=0$. Let $G(S,T) = -\dfrac{1}{2}Trace\, ST$ be the standard metric on the space $so(V)$. Then the metric h and the complex structure $\mathcal{J}$ of $T_F F_k(V,g)$ are given by:

$$h(P,Q) = 2G(P,Q) - G(FPF,Q),\ \ \mathcal{J}Q = FQ - QF + FQF^2$$

(the complex structure $\mathcal{J}$ coincides with the one defined in [11]). It is not hard to compute (see [7]) that the scalar curvature of h is equal to $\frac{1}{2}(n-k-1)(2nk-3k^2-k)$.

Now suppose that V is of odd dimension $n=2k+1$. A (linear) almost contact metric structure on the Euclidean space (V,g) is a pair (φ,ξ) of an endomorphism φ and a unit vector ξ of V such that $\varphi^2 x = -x + g(x,\xi)\xi$ and $g(\varphi x,\varphi y) = g(x,y) - g(x,\xi)g(y,\xi)$, $x,y\in V$.

Denote the set of these structures by $C(V,g)$. It is easy to see (cf. e.g. [3]) that if $(\varphi,\xi)\in C(V,g)$, then φ is a compatible partially complex structure of rank $2k$ and $\varphi(\xi)=0$. Conversely, if φ is a compatible partially complex structure of rank $2k$ and $\varphi(\xi)=0$ for a unit vector ξ, then $(\varphi,\xi)\in C(V,g)$.

The set $C(V,g)$ is a compact submanifold of $so(V)\times V$; its tangent space at a point $\sigma = (\varphi,\xi)$ consists of all pairs $(Q,\varphi Q(\xi))$ with $Q\in T_\varphi F_k(V,g)$. The group $O(V)$ of orthogonal transformations of V acts transitively on $C(V,g)$ in an obvious way and $C(V,g)$ has the homogeneous representation $C(V,g) = O(2k{+}1)/(U(k){\times}\{1\})$. Thus we have an obvious two-fold covering map

$$C(V,g) = O(2k+1)/(U(k)\times\{1\}) \to F_k(V,g) = O(2k+1)/(U(k)\times\{1,-1\}).$$

In fact this is the projection map $(\varphi,\xi)\to\varphi$. We lift the Kähler-Einstein structure of $F_k(V,g)$ to the manifold $C(V,g)$ by means of this map and denote the lifted structure again by $(h,\mathcal{J})$.

3. The twistor space $(\mathcal{C},h_t)$ and its Ricci tensor

First we recall the definition of the twistor space of partially complex structures [11].

Let (M,g) be a Riemannian manifold of dimention $n\geq 3$. Denote by $\pi:\mathcal{F}_k\to M$ the bundle over M whose fibre at a point $p\in M$ consists of all compatible partially complex structures of rank $2k$ on the Euclidean space (T_pM,g_p). This is the associated bundle

$$O(M)\times_{O(n)} F_k(\mathbb{R}^n)$$

where $O(M)$ denotes the principal bundle of orthonormal frames on M.

As it is usual in the twistor theory, the manifold $\mathcal{F}_k$ admits two partially complex structures Φ_1 and Φ_2 defined as follows [11]: (recall that a partially complex structure on a manifold is an endomorphism Φ of its tangent bundle having constant rank and such that $\Phi^3+\Phi=0$): The Levi-Civita connection on M gives rise to a splitting $\mathcal{V}\oplus\mathcal{H}$ of the tangent bundle of any bundle associated to $O(M)$ into vertical and horizontal parts. The vertical space $\mathcal{V}_f$ of $\mathcal{F}_k$ at a point $f\in\mathcal{F}_k$ is the tangent space at f of the fibre through this point and $\Phi_1|\mathcal{V}_f$ is defined to be the complex structure $\mathcal{J}_f$ of the fibre while $\Phi_2|\mathcal{V}_f$ is defined as the conjugate complex structure, i.e. $\Phi_2|\mathcal{V}_f=-\mathcal{J}_f$. The horizontal space $\mathcal{H}_f$ is isomorphic via the differential π_{*f} to the tangent space T_pM, $p=\pi(f)$, and both $\Phi_1|\mathcal{H}_f$, $\Phi_2|\mathcal{H}_f$ are defined to be the lift to $\mathcal{H}_f$ of the endomorphism f of T_pM.

The metrics of $F_k(\mathbb{R}^n)$ and M yield a 1-parameter family $h_t, t>0$, of Riemannian metrics on $\mathcal{F}_k$ such that $h_t|\mathcal{V}_f$ is t times the metric h of the fibre through f, $h_t|\mathcal{H}_f=\pi^*g$, and the spaces $\mathcal{V}_f$ and $\mathcal{H}_f$ are orthogonal. The endomorphisms Φ_1 and Φ_2 are skew-symmetric with respect to the metrics h_t and the projection $\pi:(\mathcal{F}_k,h_t)\to(M,g)$ is a Riemannian submersion with totally geodesic fibres (by the Vilms theorem).

Now let (M,g) be a Riemannian manifold of odd dimension $n=2k+1$, $k\geq 1$. Slightly modify the twistor construction in [8, 11], we define the twistor space of (M,g) as the bundle $\mathcal{C}$ over M whose fibre at a point $p\in M$ consist of all almost contact metric structures on the Euclidean space (T_pM,g_p), i.e.

$$\mathcal{C}=O(M)\times_{O(2k+1)}C(\mathbb{R}^{2k+1}).$$

Using the Levi-Civita connection on M, we can define on $\mathcal{C}$ a 1-parameter family $h_t, t>0$, and two partially complex structures Φ_1, Φ_2 of rank k^2+3k, skew-symmetric with respect to h_t in the same way as we did it for the space $\mathcal{F}_k$. Define a vector field χ on $\mathcal{C}$ by setting

$$\chi_\sigma=\xi^h_\sigma,\ \sigma=(\varphi,\xi),$$

where ξ^h_σ is the horizontal lift of ξ at the point σ. Then (Φ_α,χ,h_t), $\alpha=1,2$, is an almost contact metric structure on $\mathcal{C}$. The contact distribution of this structure is obviously $\mathcal{V}\oplus\{A\in\mathcal{H}:A\perp\chi\}$.

If the manifold M is oriented, then the twistor space $\mathcal{C}$ is the disjoint union of the open subsets $\mathcal{C}_\pm$ consisting of the points (φ,ξ) that yield $\pm$ the orientation of T_pM via the decomposition $T_pM=Im\varphi\oplus\mathbb{R}\xi$ in which the vector space $Im\varphi$ is oriented by means of the complex structure φ on it. The bundles $\mathcal{C}_\pm$ are isomorphic by the map $(\varphi,\xi)\to(\varphi,-\xi)$ which preserves the horizontal spaces and sends (Φ_α,χ,h_t) to $(\Phi_\alpha,-\chi,h_t)$.

The natural covering map $C(\mathbb{R}^{2k+1})\to F_k(\mathbb{R}^{2k+1})$ is $O(2k+1)$-equivariant, so it determines a bundle map

$$\mathcal{C}\to\mathcal{F}_k,\ (\varphi,\xi)\to\xi,$$

which is a two-fold covering. This map preserves the vertical and horizontal spaces, the metrics and the partially complex structures of $\mathcal{C}$ and $\mathcal{F}_k$. If M is oriented, each of the spaces $\mathcal{C}_+$ and $\mathcal{C}_-$ is isomorphic to $\mathcal{F}_k$.

The curvature of the Riemannian manifold $(\mathcal{F}_k, h_t)$ has been computed in [7] by means of the O'Neill formulas. The computation there immediately gives the curvature of the manifold $(\mathcal{C}, h_t)$ since the map $(\varphi, \xi) \to \xi$ above is a Riemannian covering. To formulate the corresponding result for the curvature of $(\mathcal{C}, h_t)$, we shall introduce some notations. Denote by $A(TM)$ the bundle of skew-symmetric endomorphisms of TM and consider $\mathcal{C}$ as a submanifold of the bundle $\pi : A(TM) \oplus TM \to M$. Then the inclusion of $\mathcal{C}$ is fibre-preserving and the horizontal subspace of $T_\sigma\mathcal{C}$ at a point $\sigma = (\varphi, \xi)$ coincides with the horizontal space of $A(TM) \oplus TM$ at that point. The vertical space of $\mathcal{C}$ at σ, considered as a subspace of the vertical space of $A(TM) \oplus TM$ at σ, consists of all pairs $(Q, \varphi Q(\xi))$ for which $Q \in A(T_pM)$, $p = \pi(\sigma)$, and satisfies the identity $Q\varphi^2 + \varphi^2 Q + \varphi Q \varphi + Q = 0$. Further we shall freely make use of the standard isometric identification $A(TM) \cong \Lambda^2 TM$ that assigns to each $a \in A(T_pM)$ the 2-vector $a^\wedge$ for which $g(aX, Y) = g(a^\wedge, X \wedge Y), X, Y \in T_pM$ (the metric on $\Lambda^2 TM$ is given by $g(X_1 \wedge X_2, X_3 \wedge X_4) = g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3)$). If $Q \in T_\varphi F_k(T_pM, g_p) \subset A(T_pM)$, the element $Q^\wedge$ of $\Lambda^2 T_pM$ will be also denoted by Q. For brevity, denote by m_φ the image of the tangent space $T_\varphi F_k(T_pM, g_p)$, $p = \pi(\sigma)$, under the identification $A(T_pM) \cong \Lambda^2 T_pM$. Finally, let $\mathcal{R} : \Lambda^2 TM \to \Lambda^2 TM$ be the curvature operator of (M, g); it is defined by $g(\mathcal{R}(X_1 \wedge X_2), X_3 \wedge X_4) = g((\nabla_{[X_1,X_2]} - [\nabla_{X_1}, \nabla_{X_2}])X_3, X_4))$. Now [7, Proposition 2] implies the following

Proposition 1. *Let (M, g) be a Riemannian manifold of odd dimension $n = 2k+1$, $k \geq 1$. Then the the Ricci tensor c_t of its twistor space $(\mathcal{C}, h_t)$ is given as follows: For any $E \in T_\sigma\mathcal{C}$, $\sigma = (\varphi, \xi)$, setting $X = \pi_* E$, $(Q, \varphi Q(\xi)) = \mathcal{V}E$ (the vertical component of E), we have:*

$$\begin{aligned} c_t(E, E) &= c_M(X, X) - 2tTrace(Z \to (\nabla_Z R)(\mathcal{J}Q, X)) + \\ &\quad 2t^2 ||\mathcal{R}(\mathcal{J}Q)||_g^2 - 2t||\imath_X \circ \mathcal{R}|m_\varphi||_{h,g}^2 + \frac{1}{2}k||Q||_h^2 \end{aligned}$$

where c_M is the Ricci tensor of (M, g), $\imath_X : \Lambda^2 TM \to TM$ is the interior product and $||\cdot||_{h,g}$ is the norm of the metric on the space of linear maps $m_\varphi \to T_{\pi(\sigma)}M$ induced by the metrics h on m_φ and g on $T_{\pi(\sigma)}M$.

Corollary 1. *If (M, g) is of constant curvature ν, then the Ricci tensor c_t of $(\mathcal{C}, h_t)$ is given by*

$$c_t(E, E) = 2k\nu(1 - t\nu)||X||_g^2 + t\nu^2||\varphi X||_g^2 +$$

$$\frac{1}{2}(k + 2t^2\nu^2)||Q||_h^2 + t^2\nu^2 h(\varphi \circ Q \circ \varphi, Q)$$

where $\sigma = (\varphi, \xi) \in \mathcal{C}$, $E \in T_\sigma\mathcal{C}$, $X = \pi_* E$ and $(Q, \varphi Q(\xi)) = \mathcal{V}E$.

4. Proof of the Theorem

Suppose that $(\mathcal{C}, h_t, \chi)$ is eta-Einstein. Then, by Proposition 1, there exist smooth functions a and b on $\mathcal{C}$ such that for every point $\sigma = (\varphi, \xi) \in \mathcal{C}$, every h-orthonormal basis $\{U_\alpha\}$ of $T_\varphi F_k(T_p, g_p)$, $p = \pi(\sigma)$, and every $X \in T_p M$, $Q \in T_\varphi F_k(T_p, g_p)$ the following two equations are satisfied

$$c_M(X,X) - 2t \sum_{\alpha=1}^{k^2+k} ||R(U_\alpha)X||^2 = a(\sigma)||X||^2 + b(\sigma)g(X,\xi)^2 \tag{2}$$

$$2t^2||\mathcal{R}(Q)||_g^2 + \frac{1}{2}k||Q||_h^2 = a(\sigma)t||Q||_h^2 \tag{3}$$

Lemma 1. *The functions a and b on $\mathcal{C}$ descend to smooth functions $\bar{a}$ and $\bar{b}$ on M.*

Proof. We have

$$c_t(E', E'') = ah_t(E', E'') + bh_t(E', \chi)h_t(E'', \chi) \tag{4}$$

for every $E', E'' \in T\mathcal{C}$.

Take a point $\sigma = (\varphi, \xi) \in \mathcal{C}$ and set $p = \pi(\sigma)$. Let $e_1, ..., e_{2k+1}$ be an orthonormal basis of $T_p M$ with $e_{2k+1} = \xi$ and let $V_1, ..., V_{k^2+k}$ be a h_t-orthonormal basis of the vertical space $\mathcal{V}_\sigma$ of $\mathcal{C}$. Denote by $\{A_i\}$ the h_t-orthonormal basis $\{e_1^h, ..., e_{2k+1}^h, V_1, ..., V_{k^2+k}\}$ of $T_\sigma \mathcal{C}$. Denote by D the Levi-Civita connection of the metric h_t on $\mathcal{C}$. Then a standard application of the differential Bianchi identity gives:

$$\sum_i A_m(c_t(A_i, A_i)) = \sum_{i,j} h_t((D_{A_m} R_t)(A_i, A_j, A_i), A_j) = 2\sum_i A_i(c_t(A_i, A_m))$$

where R_t is the curvature tensor of the metric h_t on $\mathcal{C}$. For every $l = 1, ..., 2k+1$ and $\beta = 1, ..., k^2+k$, we have $c_t(e_l^h, V_\beta) = 0$ by (4). Therefore

$$(k^2+3k+1)V_\beta(a) + V_\beta(b) = \sum_i V_\beta(c_t(A_i, A_i)) = 2\sum_{\gamma=1}^{k^2+k} V_\gamma(c_t(V_\gamma, V_\beta)) = 2V_\beta(a).$$

It follows that the function $(k^2+3k-1)a + b$ is constant on the fibers of $\mathcal{C}$. Denote by s the scalar curvature of (M, g). Then (2) and (3) imply that

$$s(p) + t^{-1}(k^3 + k^2) = (2k^2 + 4k + 1)a(\sigma) + b(\sigma).$$

Thus, the function $(2k^2 + 4k + 1)a + b$ is also constant on the fibers of $\mathcal{C}$. This proves the lemma. $\square$

Let $p \in M$ and let $e_1, e_2, ..., e_n$ be an orthonormal basis of $T_p M$, $n = 2k+1$. Set $e_{ij} = e_i \wedge e_j$. Assume that $n = 2k+1 \geq 5$.

Lemma 2. *For any $X \in T_pM$ we have*

$$c_M(X,X) - t\sum_{p=1}^{k}\sum_{j=2p+1}^{n}(||R(e_{2p-1,j})X||^2 + ||R(e_{2p,j})X||^2) = \bar{a}(p)||X||^2 + \bar{b}(p)g(X,e_n)^2. \tag{5}$$

Proof. Consider the point $\sigma = (\varphi,\xi) \in \mathcal{C}$ defined by $\varphi = e_{12} + ... + e_{2k-1,2k}$ and $\xi = e_{2k+1}$. Set

$$A_{pq}^{'} = \frac{1}{\sqrt{2}}(e_{2p-1,2q-1} - e_{2p,2q}),\ \ A_{pq}^{''} = \frac{1}{\sqrt{2}}(e_{2p-1,2q} + e_{2p,2q-1})$$

$$B_r^{'} = \frac{1}{\sqrt{2}}e_{2r-1,2k+1},\ \ B_r^{''} = \frac{1}{\sqrt{2}}e_{2r,2k+1}$$

where $p = 1,...,k-1$, $q = p+1,...,n$, $r = 1,...,k$. Then $\{U_\alpha\} = \{A_{pq}^{'}, A_{pq}^{''}, B_{rs}^{'}, B_{rs}^{''}\}$ is a h-orthonormal basis of $T_\varphi F_k(T_pM, g_p)$ (such that $\mathcal{J}A_{pq}^{'} = A_{pq}^{''}$, $\mathcal{J}B_{rs}^{'} = B_{rs}^{''}$). Writing (2) for this basis we get an identity which involves the basis $e = (e_1,...,e_n)$; in view of Lemma 1, its right-hand side depends on the choice of the vector e_n and does not depend on the particular choice of φ. We denote by $(2)_e$ the identity we obtain in this way. Set $e^{'} = (-e_1, e_2, ..., e_n)$. Then the identity $(2)_e - (2)_{e^{'}}$ reads as

$$\begin{aligned} &g(R(e_{13})X, R(e_{24})X) - g(R(e_{14})X, R(e_{23})X) + ...+ \\ &g(R(e_{1,2k-1})X, R(e_{2,2k})X) - g(R(e_{1,2k})X, R(e_{2,2k-1})X) = 0. \end{aligned} \tag{6}$$

Now we apply (6) for the bases $e = (e_1,...,e_4,...,e_n)$ and $e^{''} = (e_1,...,-e_4,...,e_n)$. Then the identity $(6)_e - (6)_{e^{''}}$ gives

$$g(R(e_{13})X, R(e_{24})X) - g(R(e_{14})X, R(e_{23})X) = 0. \tag{7}$$

It follows that $g(R(e_{2p-1,2q-1})X, R(e_{2p,2q})X) = g(R(e_{2p-1,2q})X, R(e_{2p,2q-1})X)$ for $p = 1,2,...,k-1$, $q = p+1, p+2, ..., n$. The latter identity, $(2)_e$ and Lemma 1 imply Lemma 2. □

The proofs of the next two lemmas go in the same lines as the proofs of Lemmas 6 and 7 in [7] (in view of Lemmas 1 and 2 above) and will be omitted.

Lemma 3. *If $i,j,,l,m \in \{1,...,n\}$ are four different indexes, then*

$$g(R(e_{ij})X, R(e_{lm})X) = 0$$

for any $X \in T_pM$.

Lemma 4. *For any $i \neq j$, $l \neq m$ and $X \in T_pM$, we have*

$$||R(e_{ij})X|| = ||R(e_{lm})X||.$$

Proof of the theorem in the case $n = 2k+1 \geq 5$.

Assume that $(\mathcal{C}, h_t, \chi)$ is eta-Einstein. Let $p \in M$ and let $e_1, ..., e_n$, $n = 2k+1$, be an orthonormal basis of T_pM. Set $\varphi = e_{12} + ... + e_{2k-1,2k}$ and $\xi = e_{2k+1}$. Set also

$$Q_1 = e_{13} - e_{24},\ Q_2 = e_{1,2k+1}.$$

Then $Q_1, Q_2 \in T_\varphi F_k(T_pM, g_p)$ and according to (3) and Lemma 1

$$||\mathcal{R}(Q_1)||^2 = ||\mathcal{R}(Q_2)||^2 = t^{-2}[\bar{a}(p)t - \frac{1}{2}k].$$

On the other hand, by Lemmas 3 and 4

$$||R(Q_1)X||^2 = 2||R(e_{13})X||^2 = 2||R(e_{1,2k+1})X||^2 = 2||R(Q_2)X||^2$$

for any $X \in T_pM$. Therefore $||\mathcal{R}(Q_1)||^2 = 2||\mathcal{R}(Q_2)||^2$ and we see that $\mathcal{R}(Q_1) = \mathcal{R}(Q_2) = 0$. Then

$$||\mathcal{R}(e_{13})|| = 2^{-1/2}||\mathcal{R}(Q_1)|| = 0 \ \text{ and } \ \bar{a}(p)t - \frac{1}{2}k = 0.$$

The first of these identities implies $R = 0$. Now taking $X = e_1$ in (2) we see that $\bar{a}(p) = 0$ which contradicts to the second identity above.

Proof of the theorem in the case $n = 3$.

Let c_M be the Ricci tensor of (M, g) and $\rho : T_pM \to T_pM$ the symmetric operator corresponding to c_M.. Denote by s the scalar curvature of (M, g). It is well-known that the curvature operator of a 3-dimensional Riemannian manifold is given by

$$\mathcal{R}(X \wedge Y) = -\frac{s}{2}X \wedge Y + \rho(X) \wedge Y + X \wedge \rho(Y) \tag{8}$$

for $X, Y \in TM$ (see e.g. [2, Sec. 1 G]). Let $p \in M$ and put

$$\lambda = \frac{1}{2t^2}[\bar{a}(p)t - \frac{1}{2}k].$$

Let e_1, e_2, e_3 be an arbitrary orthonormal basis of T_pM. Consider the point $\sigma = (\varphi, \xi) \in \mathcal{C}$ with $\varphi = e_{12}$, $\xi = e_3$. Then e_{13}, $e_{23} \in T_\varphi F_1(T_pM, g_p)$ and by (3) we have

$$||\mathcal{R}(e_{13})||^2 = ||\mathcal{R}(e_{23})||^2 = 2\lambda \ \text{ and } \ g(\mathcal{R}(e_{13}), \mathcal{R}(e_{23})) = 0.$$

It follows that either $\lambda = 0$ or the operator $T = (1/\sqrt{2\lambda})\mathcal{R}$ is orthogonal. If $\lambda = 0$, then $\mathcal{R} = 0$ and identity (2) implies $\bar{a}(p) = 0$. This together with $\lambda = 0$ gives $k = 0$, a contradiction. Thus, the operator T is orthogonal. Since this operator is also symmetric, its square is equal to Id. Therefore the eigenvalues of T are $+1$ or -1. Suppose that both $+1$ and -1 are eigenvalues of the operator T and denote by α and β the dimensions of the corresponding eigenspaces. Then

$$\frac{s}{2} = Trace\, \mathcal{R} = (\alpha - \beta)\sqrt{2\lambda}. \tag{9}$$

Further, by (2) and (3), we have that

$$s - 8t\lambda = 3\bar{a} + \bar{b},$$

therefore

$$\lambda = \frac{2ts - 3 - 2\bar{b}t}{28t^2}. \tag{10}$$

Set $c_{ij} = c_M(e_i, e_j)$. Now take the point $\sigma = (e_{13}, e_2) \in \mathcal{C}$ and apply (2) with $U_1 = \frac{1}{\sqrt{2}}e_{12}$, $U_2 = \frac{1}{\sqrt{2}}e_{23}$, $X = e_1$. This gives

$$c_{11} - t(||R(e_{12})e_1||^2 + ||R(e_{23})e_1||^2) = \bar{a}(p). \tag{11}$$

Similarly, considering the point $(e_{23}, e_1) \in \mathcal{C}$ and applying (2), we get

$$c_{11} - t(||R(e_{12})e_1||^2 + ||R(e_{13})e_1||^2) = \bar{a}(p) + \bar{b}(p).$$

Hence, $\bar{b}(p) = t(||R(e_{23})e_1||^2 - ||R(e_{13})e_1||^2)$, which, in view of (8), implies

$$\bar{b}(p) = [c_{12}^2 - (\frac{s}{2} - c_{22})^2].$$

Similarly, we have also

$$\bar{b}(p) = [c_{23}^2 - (\frac{s}{2} - c_{33})^2] \text{ and } \bar{b}(p) = [c_{13}^2 - (\frac{s}{2} - c_{11})^2].$$

Adding the last three equalities and setting $\mu = c_{12}^2 + c_{23}^2 + c_{13}^2$, we obtain

$$3\bar{b}(p) = t[\mu - (\frac{s}{2} - c_{11})^2 - (\frac{s}{2} - c_{22})^2 - (\frac{s}{2} - c_{33})^2]. \tag{12}$$

Next, an obvious application of (2) for the point $(e_{12}, e_3) \in \mathcal{C}$, gives

$$c_{11} - t(||R(e_{13})e_1||^2 + ||R(e_{23})e_1||^2) = \bar{a}(p).$$

From the latter identity and (11) we see that

$$||R(e_{12})e_1|| = ||R(e_{13})e_1||.$$

This and (8) imply

$$(-\frac{s}{2} + c_{11} + c_{22})^2 = (-\frac{s}{2} + c_{11} + c_{33})^2$$

which gives $c_{11}(c_{22} - c_{33}) = 0$. Similarly, $c_{22}(c_{33} - c_{11}) = 0$ and $c_{33}(c_{11} - c_{22}) = 0$. It follows that either $c_{11} = c_{22} = c_{33}$ or two of the numbers c_{11}, c_{22}, c_{33} are equal to 0 and the third one is different from zero.

1). Suppose $c_{11} = c_{22} = c_{33}$. Then, by (12),

$$3\bar{b}(p) = t\mu - \frac{ts^2}{12}.$$

Now it follows from (9) and (10) that ts satisfies the equation

$$(7 - \frac{m}{9})x^2 - 4mx + 6m + \frac{4mt^2\mu}{3} = 0 \tag{13}$$

where $m = (\alpha - \beta)^2$. This fact implies

$$4m^2 \geq (7 - \frac{m}{9})(6m + \frac{4mt^2\mu}{3}) \geq (7 - \frac{m}{9})6m.$$

Thus we see that $m \geq 9$. On the other hand $|\alpha - \beta| < dim\, \Lambda^2 T_pM = 3$, so $m < 9$, a contradiction.

2). Assume that $c_{11} = c_{22} = 0$. In this case, according to (12), we have

$$3\bar{b}(p) = t\mu - \frac{3ts^2}{4}$$

and, in view of (9) and (10), ts satisfies the equation

$$6x^2 - 4mx + 6m + \frac{4mt^2\mu}{3} = 0.$$

This implies $m \geq 9$ and we come again to a contradiction.

It follows that either $T = Id$ or $T = -Id$ on the whole space $\Lambda^2 T_pM$. Therefore the sectional curvature of M at each point p is constant and the classical Schur theorem implies that M is of constant curvature, say ν. Moreover ts satisfies equation (13) with $m = 9$ and $\mu = 0$. Thus $ts = 3$, i.e. $t\nu = \frac{1}{2}$.

Conversely, suppose that M is a 3-dimensional Riemannian manifold of positive constant curvature ν and take $t = \frac{1}{2\nu}$. Let $\sigma = (\varphi, \xi) \in \mathcal{C}$, $E \in T_\sigma\mathcal{C}$, $X = \pi_* E$ and $\mathcal{V}E = (Q, \varphi Q(\xi))$. Since $dim\, M = 3$, we have $\varphi \circ Q \circ \varphi = 0$ and Corollary 1 gives

$$c_t(E,E) = (2\nu - t\nu^2)||X||^2 + \frac{1}{2t}(1 + 2t^2\nu^2)||Q||^2_{h_t} - t\nu^2 g(X,\xi)^2 =$$

$$\frac{3\nu}{2}||E||^2_{h_t} - \frac{\nu}{2}g(X,\xi)^2.$$

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